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ON THE CONJECTURE OF HAJÓS

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Hajós conjectured that every s-chromatic graph contains a subdivision of K_s, the complete graph on s vertices. Catlin disproved this conjecture. We prove that almost all graphs are counter-examles in a very strong sense.

Let $G=G(n)$ be a graph of n vertices. Let $\chi=\chi(G)$ denote its chromatic number and $\sigma=\sigma(G)$ the largest integer l so that G contains a subdivision of K_l i.e. $\sigma(G)=l$ is the largest integer such that G contains a subgraph homeomorphic with complete graph of l vertices. Let us put $H(G)=\dfrac{\chi(G)}{\sigma(G)}$ and $H(n)=\max\limits_{G(n)} H(G(n))$

Hajós [10] conjectured that $H(n)=1$ and Catlin, [2] recently disproved the conjecture.

We shall prove that there are arbitrarily large graphs G with $H(G)\geqq \geqq\dfrac{\sqrt{n/2}}{(2\log n-1)^{3/2}}$. The proof is a simple consequence of Turán's Theorem and the lower bounds for Ramsey numbers $R(n, n)$ established in [6] by the probabilistic method. By a slightly more complicated method we shall prove that there is an absolute constant C such that

$$(1)\qquad H(n)>C\frac{\sqrt{n}}{\log n}$$

and in fact our proof yields that (1) holds for almost all graphs $G(n)$, i.e. (1) holds true for all but $o(2^{\binom{n}{2}})$ labelled graphs of n vertices. By a slight modification of the proof of Theorem 2 one can obtain a simpler proof of an easier fact, namely that the Hajós conjecture fails for almost all graphs.

It is difficult to guess whether probabilistic methods can be applied to disprove the Conjecture of Hadwiger. In fact — perhaps this conjecture is true after all. Various relationships between the Hajós conjecture, the Four colour Theorem and the Hadwiger Conjecture are discussed in [11].

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Catlin pointed out to the second author that graphs described in [8] in a much more complicated manner are isomorphic to some of his counterexamples to the Hajós Conjecture.

Let α and ω denote respectively the independence and the clique number.

Theorem 1. $H(G) > \frac{1}{\alpha}\sqrt{\frac{n}{2\omega}}$.

Lemma. *If G contains no q-element complete subgraphs K_q, then $\sigma(G) < \sqrt{2(q-1)n}$.*

Proof. Since G contains a subgraph homeomorphic to K_σ, there is a σ-element set $S \subseteqq G$ such that every two vertices of G are joined by mutually internally disjoint paths. Since G contains no K_q, by Turán's Theorem S has at most $\frac{q-2}{2(q-1)}\sigma^2$ edges, and thus it has at least $\beta = \frac{\sigma(\sigma-1)}{2} - \frac{q-2}{2(q-1)}\sigma^2$ missing edges. Since the endpoints of a missing edge are joined by a path of length at least two, G has at least $\beta+\sigma$ vertices (if all internal vertices of a connecting path are in S one needs even more additional paths). Thus

$$n \geqq \frac{\sigma(\sigma-1)}{2} - \frac{q-2}{2(q-1)}\sigma^2 + \sigma$$

i.e.

$$\frac{1}{2(q-1)}\sigma^2 < n$$

which proves the Lemma. ∎

Theorem 1 now follows from the Lemma since $\chi \geqq n/\alpha$. ∎

Theorem 2. *There are arbitrarily large graphs G such that*

$$H(G) \geqq \frac{\sqrt{n/2}}{(2\log n - 1)^{3/2}}.$$

Proof. By Erdős' Theorem, ([6], p. 292) for every $k > 3$ there are graphs G with more than $2^{k/2}$ vertices, containing no K_k nor any k-element independent sets. Hence, both the clique and the independence number of G are smaller than $2\log n - 1$. Thus, Theorem 2 follows from Theorem 1. ∎

Theorem 3. *There is a constant C such that for almost all graphs G,*

$$H(G) > C\frac{\sqrt{n}}{\log n}.$$

Proof. It is known, [5] that for almost all graphs G on n vertices we have

$$\chi(G) > C_1 \frac{n}{\log n}. \tag{2}$$

Thus to prove the theorem it is enough to show that for almost all graphs $G(n)$, we have

(3) $$\sigma(G) < C_2\sqrt{n}.$$

From the Central Limit Theorem (or an elementary combinatorial computation) it follows that the number of graphs of t vertices which have more than $\frac{2}{3}\binom{t}{2}$ edges is less than $2^{\binom{t}{2}} \cdot e^{-ct^2}$. Hence the number of graphs on n vertices which have a subgraph of $t > C_3 \log n$ vertices which has more than $\frac{2}{3}\binom{t}{2}$ edges is less than

$$2^{\binom{n}{2}}\binom{n}{t}e^{-ct^2} < 2^{\binom{n}{2}}n^t e^{-ct^2} = o\left(2^{\binom{n}{2}}\right).$$

Thus, almost all graphs on n vertices have the property that for every $t > C_3 \log n$, every subgraph of t vertices misses at least $\frac{1}{3}\binom{t}{2}$ edges. Hence, by the same argument as in the Lemma we have that all but $o(2^{\binom{n}{2}})$ graphs have $\sigma(G) < C_2\sqrt{n}$, which together with (2) proves the theorem. ▌

The proof of Theorem 3 could be easily improved to show that for almost all graphs $G(n)$ we have

$$\sigma(G(n)) < (2+o(1))n^{1/2}.$$

We also conjecture that

$$H(n) < C\frac{n^{1/2}}{\log n},$$

i.e. that our theorem is best possible apart from the value of the constant.

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